On the Limits of Recursive Characterizations in the Refined A-Translation

Constructive and Synthetic Methods in Algebra, Geometry, and Foundations of Mathematics

Franziskus Wiesnet

TU Wien

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Weakness:

- $\bot$ can be part of A.
- There may be some assumptions about $\bot$ in M.

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- Define formula classes $\mathcal{D}, \mathcal{G}$ with $D^{\mathbf{F}} \to D$ and $G \to (G^{\mathbf{F}} \to \bot) \to \bot$ for all $D \in \mathcal{D}$ and $G \in \mathcal{G}$.

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- Then $D \to \forall_x (G \to \perp) \to \perp$ implies $D^{\mathbf{F}} \to \exists_x G^{\mathbf{F}}$.

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Definition (Types and Terms in NA^ω, MA^ω, HA^ω and PA^ω)

We fix an infinite set of type variables denoted by ξ, ζ. Types are syntactically defined by

$$\tau, \rho ::= \xi \mid \mathbb{B} \mid \mathbb{N} \mid \mathbb{L}(\tau) \mid \tau \to \rho \mid \tau \times \rho.$$

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For each type τ we fix a countable set of object variables denoted by x^τ, y^τ, z^τ. Furthermore, we have the usual typed constructors and destructors such as tt, ff, Cases^τ 0, S, nil_τ, $\cdot ::_\tau \cdot$, $\mathcal{R}^\tau_\alpha$ and so on.

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- Each object variable, constructor and destructor is a term with its type given above.
- If $t : \tau \to \rho$ and $s : \tau$ are terms, then also $ts : \rho$ is a term.
- If $x : \tau$ is an object variable and $t : \rho$ is a term, then $\lambda_x t : \tau \to \rho$ is a term.

We denote the set of all terms by Term.

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- For each boolean term $t : \mathbb{B}$ the atomic formula $\mathrm{at}(t)$ is a formula.
- If A and B are formulas, then so are $A \to B$ and $A \wedge B$.
- If A is a formula and x is an object variable, then $\forall_x A$ is a formula.

Abbreviations: $\mathbf{T} := \mathrm{at}(\mathrm{tt})$, $\mathbf{F} := \mathrm{at}(\mathrm{ff})$, $\neg A := A \to \mathbf{F}$.

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To define formulas in MA^ω we fix a new symbol $\bot$ and add the clause that $\bot$ is also a formula.

To define formulas in HA^ω and PA^ω we add the following rules to NA^ω:

- If A and B are formulas, then so is $A \vee B$.
- If A is a formula and x is an object variable, then $\exists_x A$ is a formula.

For $\mathrm{XA}^\omega \in \{\mathrm{NA}^\omega, \mathrm{MA}^\omega, \mathrm{HA}^\omega, \mathrm{PA}^\omega\}$ we denote the sets of all formulas in XA^ω by $\mathrm{Form}_{\mathrm{XA}^\omega}$.

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We have the usual axioms in NA^ω such as $\mathrm{Truth} : \mathbf{T}$, $\mathcal{C}^{b,A} : \forall_b(A(\mathrm{tt}) \to A(\mathrm{ff}) \to A(b))$ and so on, together with the usual clauses for $\to, \forall$ and $\wedge$

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To define proofs in MA^ω we add the axiom

$$\perp^+ : \mathbf{F} \to \perp.$$

To define proofs in HA^ω we take the axioms and rules of NA^ω and add axioms for $\vee$ and $\exists$.

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To define proofs in HA^ω we take the axioms and rules of NA^ω and add axioms for $\vee$ and $\exists$.

The theory PA^ω is then HA^ω together with the law of excluded middle

$$\mathrm{lem} : A \vee \neg A.$$

Useful Lemmas

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Lemma (Ex-falso)

For $\mathsf{XA}^\omega \in \{\mathsf{NA}^\omega, \mathsf{MA}^\omega, \mathsf{HA}^\omega, \mathsf{PA}^\omega\}$ *and any formula* $A \in \mathrm{Form}_{\mathsf{XA}^\omega}$, $\mathbf{F} \to A$ *is provable in* XA^ω.

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Lemma (Ex-falso)

For $\mathrm{XA}^\omega \in \{\mathrm{NA}^\omega, \mathrm{MA}^\omega, \mathrm{HA}^\omega, \mathrm{PA}^\omega\}$ *and any formula* $A \in \mathrm{Form}_{\mathrm{XA}^\omega}$, $\mathbf{F} \to A$ *is provable in* XA^ω.

Lemma

For $\mathrm{XA}^\omega \in \{\mathrm{HA}^\omega, \mathrm{MA}^\omega, \mathrm{NA}^\omega\}$, *let* $S \in \mathrm{Form}_{\mathrm{XA}^\omega}$ *and* $A \in \mathrm{Form}_{\mathrm{MA}^\omega}$ *and assume that* A *is provable in* MA^ω *then there is a proof of* $A^S := A[\bot := S]$ *in* XA^ω.

Gödel-Gentzen translation

Definition

For any formula $A \in \mathrm{Form}_{\mathrm{PA}^\omega}$ we define its Gödel-Gentzen translation $A^{\neg\neg}$ recursively by:

$$
\begin{aligned}
P^{\neg\neg} &:= \neg\neg P \quad \text{for atomic } P \\
(A \circ B)^{\neg\neg} &:= A^{\neg\neg} \circ B^{\neg\neg} \quad \text{for } \circ \in \to, \wedge \\
(\forall_x A)^{\neg\neg} &:= \forall_x A^{\neg\neg}, \\
(A \vee B)^{\neg\neg} &:= \neg(\neg A^{\neg\neg} \wedge \neg B^{\neg\neg}), \\
(\exists_x A)^{\neg\neg} &:= \neg\forall_x \neg A^{\neg\neg},
\end{aligned}
$$

Theorem

Let A be any statement in Peano arithmetic, then

$$\mathrm{PA}^\omega \vdash A \quad \Longleftrightarrow \quad \mathrm{NA}^\omega \vdash A^{\neg\neg}.$$

Definite, Goal, Relevant and Irrelevant Formulas

Definition (Version by Schwichtenberg and Wainer, without $\wedge$)

We define the classes $\mathcal{D}, \mathcal{G}, \mathcal{R}, \mathcal{I} \subseteq \mathrm{MA}^\omega$ simultaneously as follows:

$$R, P, I \to D, \forall_x D \in \mathcal{D}$$
$$I, \bot, R \to G, D_0 \to G \in \mathcal{G}$$
$$\bot, G \to R, \forall_x R \in \mathcal{R}$$
$$P, D \to I, \forall_x I \in \mathcal{I},$$

where P is atomic, $D \in \mathcal{D}, G \in \mathcal{G}, R \in \mathcal{R}, I \in \mathcal{I}$ and $D_0 \in \mathcal{D}$ and quantifier-free.

Note that the new version is more complex to describe and therefore not given here.

Definite, Goal, Relevant and Irrelevant Formulas

Lemma

For $D \in \mathcal{D}$, $G \in \mathcal{G}$, $R \in \mathcal{R}$ and $I \in \mathcal{I}$ there are derivations of

$$
\begin{aligned}
D^{\mathbf{F}} &\to D \\
G &\to (G^{\mathbf{F}} \to \bot) \to \bot \\
((R^{\mathbf{F}} \to \mathbf{F}) \to \bot) &\to R \\
I &\to I^{\mathbf{F}}
\end{aligned}
$$

Lemma

Let $D, G \in \mathrm{Form}_{\mathrm{MA}^\omega}$ *be a formulas such that*

$$\mathrm{MA}^\omega \vdash D^{\mathbf{F}} \to D, \quad \textit{and} \quad \mathrm{MA}^\omega \vdash \forall_x (G \to (G^{\mathbf{F}} \to \bot) \to \bot).$$

Assume moreover $\mathrm{MA}^\omega \vdash D \to \forall_x (G \to \bot) \to \bot$. *Then*

$$\mathrm{HA}^\omega \vdash D^{\mathbf{F}} \to \exists_x G^{\mathbf{F}}.$$

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Theorem (Refined A-Translation)

Let $D \in \mathcal{D}$ *and* $G \in \mathcal{G}$ *be given with* $\mathrm{MA}^\omega \vdash D \to \forall_x (G \to \bot) \to \bot$. *Then*

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Answer: No. Let A be any prime formula not in $\{\bot, \mathbf{F}\}$. We define

$$S := \forall_x(\neg\neg A \to A)$$
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Then $\mathrm{MA}^\omega \vdash (S \to T)^{\mathbf{F}} \to (S \to T)$, but $S \to T \notin \mathcal{D}$.

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Then $\mathrm{MA}^\omega \vdash (S \to T)^{\mathbf{F}} \to (S \to T)$, but $S \to T \notin \mathcal{D}$.

Open problem: Find a useful characterization of the class of formulas such that $\mathrm{MA}^\omega \vdash D^F \to D$.

Solution to the open problem

Theorem

There is no recursive characterization for any of the formula classes

$$\{D \in \mathrm{Form}_{\mathrm{NA}^\omega} \mid \mathrm{MA}^\omega \vdash D^{\mathbf{F}} \to D\}, \quad \{G \in \mathrm{Form}_{\mathrm{NA}^\omega} \mid \mathrm{MA}^\omega \vdash G \to (G^{\mathbf{F}} \to \bot) \to \bot\},$$
$$\{R \in \mathrm{Form}_{\mathrm{NA}^\omega} \mid \mathrm{MA}^\omega \vdash ((R^{\mathbf{F}} \to \mathbf{F}) \to \bot) \to R\}, \quad \{I \in \mathrm{Form}_{\mathrm{NA}^\omega} \mid \mathrm{MA}^\omega \vdash I \to I^{\mathbf{F}}\}.$$

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As an example, we show that $\mathrm{MA}^\omega \vdash G \to (G^{\mathbf{F}} \to \bot) \to \bot$ is not recursively decidable.

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Let $A \in \mathrm{Form}_{\mathrm{NA}^\omega}$ and define $G := A \to \bot$.

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Assume this were the case. Let a Turing machine T be given and take $\tilde{A} \in \mathrm{Form}$ such that $\mathrm{PA}^\omega \vdash \tilde{A} \Leftrightarrow T$ terminates. (!!).

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New open problem

Recall: Let A be any prime formula not in $\{\bot, \mathbf{F}\}$. We define

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Therefore one could add all formulas of the form $S \to T$ to $\mathcal{D}$.

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$$T := (\forall_x A \to \bot) \to \bot.$$

Then $\mathrm{MA}^\omega \vdash (S \to T)^{\mathbf{F}} \to (S \to T)$, but $S \to T \notin \mathcal{D}$.

Therefore one could add all formulas of the form $S \to T$ to $\mathcal{D}$.
But there is currently no known practical application for such formulas.

New open problem

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New open question: Find a practical application of the refined A-Translation, with a proper extension of the classes $\mathcal{D}, \mathcal{G}, \mathcal{R}$ and $\mathcal{I}$.

Thank you!