

Program extraction in elementary arithmetic: Fundamental Theorem of Arithmetic and Friends

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Computational Formulas

General Examples

The elementary computationally relevant formulas **in our context** are of the form

$$A \vee B \quad \text{and} \quad \exists_x A.$$

$A \rightarrow B$ is computationally relevant iff B is.

$A \wedge B$ is computationally relevant iff at least one of the conjunction parts is computationally relevant.

$\forall_x A$ is computationally relevant iff A is.

Equalities and boolean terms are non-computational.

Examples are $\neg A$ for any formula A ;

$A \vee^b B$, $A \wedge^b B$ for boolean terms A and B , as well as

$n = \text{gcd}(n, m)$, $\exists_i^{<S} i \cdot n = m$, ...

Formal Program Extraction from Proofs

Compact Overview

Let a computationally relevant formula A be given.

From A one can calculate the **type** $\tau(A)$ and the **realiser predicate** A^r of A .

From a formal proof M of A one can construct the **extracted term** $\text{et}(M)$ of type $\tau(A)$.

If A and M are **r-free**, the **soundness theorem** states

$$A^r \text{ et}(M).$$

The construction of $\text{et}(M)$ is implemented in Minlog.

Computational and Non-Computational Formulas

Concrete Examples

- ▶ “6 is a composite number.” is non-computational ($\exists_{i,j}^{<6} i \cdot j = 6$).
- ▶ “2 is a prime number.” is non-computational ($1 < 2 \wedge \neg \exists_{i,j}^{<2} i \cdot j = 2$).
- ▶ “If n divides p , then $n = 1$ or $n = p$.” is usually defined by $n \mid p \rightarrow n = 1 \vee n = p$ and is thereby non-computational.
- ▶ “There are infinitely many prime numbers.” is computationally relevant.
- ▶ “Every natural number has a prime factorization.” is computationally relevant.
- ▶ “The prime factorization of natural numbers is unique up to the order of the factors.” is computationally relevant.

Numeric types

Nat vs. Pos

The **natural numbers** \mathbb{N} are given by

$$0 : \mathbb{N}, \quad S : \mathbb{N} \rightarrow \mathbb{N}.$$

The **positive binary numbers** \mathbb{P} are given by

$$1 : \mathbb{P}, \quad S_0 : \mathbb{P} \rightarrow \mathbb{P}, \quad S_1 : \mathbb{P} \rightarrow \mathbb{P}.$$

Note that the well-known binary representation is in reversed order. For example, $S_0 S_1 1$ represents the number 6, which corresponds to binary 110.

There are canonical functions

$$\text{PosToNat} : \mathbb{P} \rightarrow \mathbb{N} \quad \text{and} \quad \text{NatToPos} : \mathbb{N} \rightarrow \mathbb{P},$$

with $\text{NatToPos}(0) := 1$.

Least natural number

For a sequence $ws : \mathbb{N} \rightarrow \mathbb{B}$ and natural numbers $n < m$,

$$\mu_{n \leq i < m}(ws\ i)$$

denotes the smallest number $i < m$ with $ws(i)$ and $n \leq i$, if it exists, and m otherwise.

By using PosToNat and NatToPos this can also be defined for positive binary numbers.

Example:

The natural square root function is then defined by

$$\lceil \sqrt{n} \rceil := \mu_{i < S_n}(n \leq i \cdot i).$$

Interval nesting on positive binary numbers

Let $wf : \mathbb{P} \rightarrow \mathbb{B}$ and $n : \mathbb{N}$ be given, such that $wf(1)$, $\neg wf(2^n)$ and $wf(q) \rightarrow wf(p)$ for all $p < q$. The greatest p with $wf(p)$ is denoted by

$$\nu(wf, n).$$

This value is defined very efficiently by interval nesting using the binary representation.

Example:

$$\lfloor \sqrt{p} \rfloor := \nu \left(\lambda_q (q \cdot q \leq p), \left\lfloor \frac{\text{PosLog } p}{2} \right\rfloor + 1 \right)$$

GCD algorithm

Euclidean algorithm

Definition

The greatest common divisor on the natural numbers is defined by the following rules:

$$\gcd(0, n) := n$$

$$\gcd(m, 0) := m$$

$$\gcd(S m, S n) := \begin{cases} \gcd(S m, n - m) & \text{if } m < n \\ \gcd(m - n, S n) & \text{otherwise} \end{cases}$$

GCD algorithm

Stein's algorithm

Definition

The greatest common divisor on the positive binary numbers is defined by the following rules:

$$\gcd(1, p) := 1$$

$$\gcd(S_0 p, 1) := 1$$

$$\gcd(S_0 p, S_0 q) := S_0(\gcd(p, q))$$

$$\gcd(S_0 p, S_1 q) := \gcd(p, S_1 q)$$

$$\gcd(S_1 p, 1) := 1$$

$$\gcd(S_1 p, S_0 q) := \gcd(S_1 p, q)$$

$$\gcd(S_1 p, S_1 q) := \begin{cases} \gcd(S_1 p, q - p) & \text{if } p < q \\ \gcd(p - q, S_1 q) & \text{if } p > q \\ S_1 p & \text{otherwise.} \end{cases}$$

Versions of Bézout's Identity

Theorem (Bézout's identity)

For two integers $a, b \in \mathbb{Z}$ there are $u, v \in \mathbb{Z}$ with $\gcd(a, b) = ua + vb$.

Theorem (Bézout's identity on natural numbers)

$$\forall_{n,m} \exists_{l_0} \exists_{l_1}. \gcd(n, m) + l_0 \cdot n = l_1 \cdot m \vee \gcd(n, m) + l_0 \cdot m = l_1 \cdot n$$

Theorem (Positive binary version of Bézout's identity)

$$\begin{aligned} \forall_{p_0, p_1}. \quad & \exists_q \quad q \cdot p_0 = p_1 \\ & \vee \exists_q \quad q \cdot p_1 = p_0 \\ & \vee \exists_{q_0} \exists_{q_1} \quad \gcd(p_0, p_1) + q_0 \cdot p_0 = q_1 \cdot p_1 \\ & \vee \exists_{q_0} \exists_{q_1} \quad \gcd(p_0, p_1) + q_1 \cdot p_1 = q_0 \cdot p_0 \end{aligned}$$

The Fundamental Theorem of Arithmetic

Existence of the Prime Factorisation

Theorem

$$\forall p \exists ps \exists m. \mathbf{P}_m(ps) \wedge \prod_{i < m} ps(i) = p$$

Notation.

$$ps : \mathbb{N} \rightarrow \mathbb{P}$$

$$\prod_{i < 0} ps(i) := 1, \quad \prod_{i < n+1} ps(i) := \left(\prod_{i < n} ps(i) \right) \cdot ps(n)$$

The Fundamental Theorem of Arithmetic

Uniqueness of the Prime Factorisation

Theorem

$$\forall_{n,m,ps,qs} \cdot \mathbf{P}_n(ps) \rightarrow \mathbf{P}_m(qs) \rightarrow \prod_{i < n} ps(i) = \prod_{i < m} qs(i) \rightarrow \\ n = m \wedge \exists_f \exists_g (\mathbf{Pms}_n(f, g) \wedge \forall_{i < n} ps(f i) = qs(i)).$$

Notation.

$$f, g : \mathbb{N} \rightarrow \mathbb{N},$$

$$\mathbf{Pms}_n(f, g) :\Leftrightarrow f \circ g = g \circ f = \text{id} \quad \wedge \quad \forall_{i \geq n} f(i) = i$$

Fermat Factorisation

Motivation

For natural numbers $a > b$, we have

$$a^2 - b^2 = (a + b)(a - b).$$

Idea: Let n be given, search for m such that $m^2 - n$ is a square, say $m^2 - n = l^2$, then $n = (m + l)(m - l)$. If $m - l > 1$, this yields a non-trivial factorisation of n .

Question: Is this search always successful for composite numbers? And is this search bounded?

Answer: Yes, for odd numbers.

Fermat Factorisation

Lemma

Let $p = 2q + 1 > 1$ be an odd number, that is not a perfect square. If p is a composite number, then $p = p_1^2 - p_0^2$ with $p_0 < p_1 < q$.

Proof.

Let $p = q_0 \cdot q_1$ with $q_0 < q_1$. As p is odd, also q_0, q_1 must be odd, hence

$$q_0 = 2r_0 + 1 \quad \text{and} \quad q_1 = 2r_1 + 1.$$

We define

$$p_0 := \frac{q_1 - q_0}{2} = r_1 - r_0 > 0, \quad p_1 := \frac{q_1 + q_0}{2} = r_1 + r_0 + 1 > 0.$$

Then

$$p_1^2 - p_0^2 = q_0 \cdot q_1 = p.$$

Clearly, $p_0 < p_1$. Furthermore, $q_0, q_1 > 2$ and $q_0, q_1 < q$, therefore $p_1 < q$.



Fermat Factorisation

Theorem

Let $p > 1$ be a natural number. Then p is either prime or there are $q_0, q_1 > 1$ with $p = q_0 \cdot q_1$.

Proof.

Without loss of generality, we may assume that $p = 2q + 1$ is odd, not a perfect square, and $p \notin \{3, 5\}$.

From $p > 5$ we get $\lfloor \sqrt{p} \rfloor < q$ (!), and therefore define

$$l := \mu_{\lfloor \sqrt{p} \rfloor \leq i < q}(\text{IsSq}(i^2 - p)).$$

If $l = q$ there is not $i < q$ with $\text{IsSq}(i^2 - p)$ and therefore p is prime by the lemma above. If $l < q$, we have $l^2 - p = r^2$ for $r = \lfloor \sqrt{l^2 - p} \rfloor$. Therefore $p = l^2 - r^2 = (l + r)(l - r)$. Furthermore $l - r \neq 1$ (!), which completes the proof. □