Material Interpretation and Constructive Analysis of Maximal Ideals in $\mathbb{Z}[X]$

Franziskus Wiesnet

TU Wien
This research was funded in whole or in part by the Austrian Science Fund (FWF) **10.55776/ESP576**.

February 25, 2026

FWF Österreichischer Wissenschaftsfonds

Material Interpretation

General Concept

Given a possibly classical proof of a statement of the form $A \to B$.

Goal: A proof for a statement $\neg A \vee B$, where $\neg A$ is a constructively stronger form of the negation of A.

A and B may also be slightly modified. However, the statement and the proof should remain as close as possible to their original form.

Case Study

A Classical Definition

Definition

Let R be a ring. An IDEAL $I \subseteq R$ is a subset with the following properties:

- $0 \in I$
- $a, b \in I \to a + b \in I$
- $\lambda \in R, a \in I \to \lambda a \in I$

We say that an ideal $M \subsetneq R$ is a MAXIMAL if for all ideal $I \subseteq R$ with $M \subseteq I$ we either have $I = M$ or $I = R$.

Classically equivalent: $M \neq R$ and for all $a \in R \setminus M$ there is $\lambda \in R$ with $\lambda a - 1 \in M$.

Case Study

A Classical Proof

Theorem

Let $M \subseteq \mathbb{Z}[X]$ be a maximal ideal. Then, there exists a prime number p with $p \in M$.

Proof.

There is some non-constant $f \in M$: Either $X \in M$, or $X \notin M$ and there is some $g \in \mathbb{Z}[X]$ with $gX - 1 \in M$ as M is maximal.

Let d be the leading coefficient of f. Assume there is no prime number p with $p \in M$. As a maximal ideal is also a prime ideal, $M \cap \mathbb{Z} = \{0\}$. Hence the canonical homomorphism $\mathbb{Z} \to \mathbb{Z}[X]/M$ is injective into the field $\mathbb{Z}[X]/M$ and induces a ring extension $\mathbb{Z}[d^{-1}] \to \mathbb{Z}[X]/M$. This is an **integral ring extension** with the integral polynomial $d^{-1}f$. As $\mathbb{Z}[X]/M$ is a field, also $\mathbb{Z}[d^{-1}]$ must be a field, which is impossible. □

Case Study

An Explicit Definition

That M is a maximal ideal was used in the proof as follows: If $f \notin M$, then there exists a $g \in \mathbb{Z}[X]$ such that $fg - 1 \in M$.

Furthermore, we used case distinction on membership in M. Strictly speaking, M was not treated as a set, but rather as a total function from $\mathbb{Z}[X]$ to $\mathbb{B} = \{0, 1\}$.

FWF Österreichischer Wissenschaftsfonds

Definition

Let R be a ring. For a boolean valued function $M : R \to \mathbb{B}$ and a function $\nu : R \to R$, we say that (M, ν) is an EXPLICIT MAXIMAL IDEAL if M is an ideal, $1 \notin M$ and $a\nu(a) - 1 \in M$ for all $a \in R \setminus M$.

Furthermore, we say that there is EVIDENCE THAT (M, ν) IS NOT AN EXPLICIT MAXIMAL IDEAL if one of the following cases holds:

- $0 \notin M$,
- there are $a, b \in M$ with $a + b \notin M$,
- there are $\lambda \in R$ and $a \in M$ with $\lambda a \notin M$,
- $1 \in M$, or
- there is $a \in R \setminus M$ with $a\nu(a) - 1 \notin M$.

Theorem

Let $M : \mathbb{Z}[X] \to \mathbb{B}$ *and* $\nu : \mathbb{Z}[X] \to \mathbb{Z}[X]$ *be given. Then, either there exists a prime number* $p \in M$*, or there is evidence that* (M, ν) *is not an explicit maximal ideal in* $\mathbb{Z}[X]$*.*

For this presentation we only show:

> *There is some non-constant* $f \in M$ *or there is evidence that* (M, ν) *is not an explicit maximal object.*

If $X \in M$, we are done. Otherwise $X \notin M$ and either $X\nu(X) - 1 \in M$ or there is evidence that (M, ν) is not an explicit maximal ideal. Either $X\nu(X) - 1$ is non constant, or $\nu(X) = 0$ and therefore $-1 \in M$. Either $1 \in M$ or $(-1) \cdot (-1) \notin M$. In both cases, there is evidence that (M, ν) is not an explicit maximal ideal.

FWF Österreichischer Wissenschaftsfonds

Case Study

Remarks

- A few “non-constructive” principles remain. In particular, we implicitly assumed that membership in M is decidable by writing $M : \mathbb{Z}[X] \to \mathbb{B}$ and not $M \subseteq \mathbb{Z}[X]$.
  That this assumption was necessary is also evident from the classical proof, which made a case distinction based on membership in M.
- The definition of an explicit maximal ideal is also derived from the classical proof, since we also needed an element g with $gX - 1 \in M$ there.
- Instead of applying Modus Ponens, there is often a case distinction if a certain element is in M or not.

Theorem (Hilbert's 17th Problem)

Let $f \in \mathbb{Q}[X_1, \ldots, X_n]$ be given with $f(\vec{x}) \geq 0$ for all $\vec{x} \in \mathbb{Q}^n$. Then f is a sum of squares in $\mathbb{Q}(X_1, \ldots, X_n)$.

The problem was classically solved in 1927 by Emil Artin[1] using several lemmas, including Sturm's theorem and the **Artin-Schreier Theorem**[2]:

Theorem (Artin-Schreier Theorem)

Let K be an field, then

$$\bigcap \{U \subseteq K \mid U \text{ is an order of } K\} = \left\{ \sum_{i=0}^{n} x_i^2 \;\middle|\; n \in \mathbb{N},\ x_0, \ldots, x_n \in K \right\}.$$

Hilbert's 17th Problem was constructively considered by Charles N. Delzell in 1984 [3].

Future Application

Theorem (Zariski's Lemma)

Let K be a field and R an K-algebra, which is also a field. Suppose that $R = K[x_1, \ldots, x_n]$ for some $x_1, \ldots, x_n \in R$. Then R is algebraic over K, i.e. there are non-zero $f_1, \ldots, f_n \in K[X]$ such that $f_i(x_i) = 0$ for all i.

This theorem could also be used to prove the statement in the case study above. In 1947 Zariski used it to prove Hilbert's Nullstellensatz [5].

Theorem (Hilbert's Nullstellensatz)

Let K be an algebraically closed field, $\vec{X} := X_1, \ldots, X_n$ and $f_1, \ldots, f_m \in K[\vec{X}]$ be given. Then, either there are $g_1, \ldots, g_m \in K[\vec{X}]$ with $g_1 f_1 + \cdots + g_m f_m = 1$ or there are $x_1, \ldots, x_n \in K$ with $f_i(x_1, \ldots, x_n) = 0$ for all i.

An algorithmic version of Zariski's Lemma was already developed, which can be used to develop a material interpretation of Zariski's Lemma [4]. This can lead to a material interpretation of Hilbert's Nullstellensatz.

Thank you!

Questions are welcome.

Emil Artin.
Über die Zerlegung definiter Funktionen in Quadrate.
Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg, 5(1):100–115, December 1927.

Emil Artin and Otto Schreier.
Algebraische Konstruktion reeller Körper.
Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg, 5(1):85–99, December 1927.

C. N. Delzell.
A continuous, constructive solution to Hilbert's 17th problem.
Inventiones Mathematicae, 76(3):365–384, October 1984.

Franziskus Wiesnet.
An Algorithmic Version of Zariski's Lemma, pages 469–482.
Lecture Notes in Computer Science. Springer International Publishing, 2021.

Oscar Zariski.
A new proof of Hilbert's Nullstellensatz.
Bulletin of the American Mathematical Society, 53:362–368, 1947.